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Yann Brenier

Stanley Osher

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Hampton, Virginia 23665

THE DISCRETE ONE-SIDED LIPSCHITZ CONDITION FOR CONVEX SCALAR CONSERVATION LAWS

Yann Brenier[1,2] and Stanley Osher[3]
Department of Mathematics, UCLA
Los Angeles, CA 90024

ABSTRACT

Physical solutions to convex scalar conservation laws satisfy a one-sided Lipschitz condition (OSLC) that enforces both the entropy condition and their variation boundedness. Consistency with this condition is therefore desirable for a numerical scheme and was proved for both the Godunov and the Lax-Friedrichs scheme--also, in a weakened version, for the Roe scheme, all of them being only first order accurate. A new, fully second order scheme is introduced here, which is consistent with the OSLC. The modified equation is considered and shows interesting features. Another second order scheme is then considered and numerical results are discussed.

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[1]Present Address: INRIA, Domaine de Voluceau - Rocquencourt, B.P. 105, 78153 Le Chesnay, France.

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1. **The OSLC for Convex Conservation Laws.** For a given (say locally integrable) function $v(x)$, it makes sense to consider the one-sided Lipschitz semi-norm:

$$p(v) = \sup_{x \neq y} \Big(\frac{v(x) - v(y)}{x - y} \Big)_+ \qquad \text{where } a_+ = \max(0, a)). \tag{1.1}$$

If v is a decreasing function, then $p(v) = 0$. Conversely, if v has increasing jumps, then $p(v) = +\infty$.

It has been known for a long time [7], that the physical solutions to a scalar conservation law:

$$u_t + f(u)_x = 0 \tag{1.2}$$

with a (strictly) convex flux function:

$$f''(u) \geq \alpha > 0, \tag{1.3}$$

satisfy:

$$\frac{u(t,x) - u(t,y)}{x - y} \leq \frac{1}{\alpha t}, \quad (\text{for } t > 0 \text{ and } x \neq y), \tag{1.4}$$

that is, in terms of the semi-norm:

$$p(u(t,\cdot)) \leq \frac{1}{\alpha t} \text{ for } t > 0. \tag{1.5}$$

In particular, the semi-norm of the solution at time $t > 0$ is always finite, even when the semi-norm of the initial data is infinite. In other words, any increasing discontinuity introduced in the initial data is immediately spread out and becomes a rarefaction wave - and then only decreasing jumps, or shock waves, can appear. This is precisely the correct entropy condition for convex conservation laws. It is therefore automatically enforced by the OSLC (1.5).

Moreover, the total variation boundedness of the solutions (a well known property of any scalar conservation law) is also enforced by the OSLC (1.5): Let us

consider, for simplicity, the initial value problem associated with (1.9), with periodic boundary conditions:

$$u(t,0) = u(t,L), \forall t > 0, \tag{1.6}$$

where $L > 0$ is the period. Then the following estimate

$$TV(u(t,\cdot)) \leq 2Lp(u(t,\cdot)) \leq 2L(\alpha t)^{-1} \tag{1.7}$$

holds, as a consequence of the following lemma:

LEMMA 1. *Let $v(x)$ be a locally integrable function of period L, then*

$$TV(v) \leq 2p(v)L. \tag{1.8}$$

Proof. The proof is given when v is smooth and can be easily generalized by a standard density argument. We have:

$$TV(v) = \int_0^L |v'(x)|dx \leq \int_0^L \{|v'(x) - p(v)| + p(v)\}dx$$

By the definition of $p(v)$, we have: $v'(x) \leq p(v)$ and thus $|v'(x)-p(v)| = p(v)-v'(x)$. Therefore: $TV(v) \leq \int_0^L [2p(v) - v'(x)]dx = 2Lp(v) - \int_0^L v'(x)dx = 2Lp(v)$ (since v is periodic). Q.E.D.

So, at least for the periodic boundary problem, (in fact, also for the Cauchy problem), the OSLC enforces both entropy consistency and total variation boundedness.

As stated in (1.5), the OSLC is not entirely satisfying, since $(\alpha t)^{-1}$ blows up when t approaches 0. A more refined estimate can be quite easily deduced for $p(u(t,\cdot))$ through the viscosity method:

PROPOSITION 1. *Any physical solution $u(t,x)$ to (1.2-3) satisfies:*

$$p(u(t,\cdot)) \le (p(u(s,\cdot))^{-1} + \alpha(t-s))^{-1} \quad \textit{for} \quad t \ge s \ge 0 \tag{1.9}$$

and in particular,

$$p(u(t,\cdot)) \le p(u(s,\cdot)), \quad \textit{for} \quad t \ge s \ge 0. \tag{1.10}$$

We do not claim that this is a new result.

Before proving this proposition, let us notice that, when $p(u(0,\cdot))$ is finite (in particular when $u(0,x)$ is smooth), then $p(u(t,\cdot))$ is uniformly bounded in time.

Proof (sketch) Let $u^\varepsilon(t,x)$ be the (presumably unique and smooth) solution to:

$$u^\varepsilon_t + f(u^\varepsilon_x) = u^\varepsilon_{xx}. \tag{1.11}$$

A trivial computation leads to:

$$\frac{d}{dt} p(u^\varepsilon(t,\cdot)) \le -\alpha p^2(u^\varepsilon(t,\cdot)) \tag{1.12}$$

that is (after integration)

$$p(u^\varepsilon(t,\cdot)) \le (p(u^\varepsilon(s,\cdot))^{-1} + \alpha(t-s))^{-1} \quad \text{for} \quad t \ge s \ge 0. \tag{1.13}$$

Then (1.10) is obtained when $\varepsilon \to 0$. Q.E.D.

II. Convergence of OSLC consistent numerical schemes. As usual, an explicit conservative numerical scheme is defined by:

$$(u_i^{n+1} - u_i^n)\Delta x + (f^n_{i+\frac{1}{2}} - f^n_{i-\frac{1}{2}})\Delta t = 0 \tag{2.1}$$

where $\Delta = (\Delta t, \Delta x)$ denotes the time and space steps, u_i^n is an approximation to $u(n\Delta t, i\Delta x)$ and $f^n_{i+\frac{1}{2}}$ is the numerical flux satisfying the usual requirements $f_{i+\frac{1}{2}}(u_{i+k},...,u_{i-k+1})$ is a Lipschitz function of $2k$ variables with

$$f_{i+\frac{1}{2}}(u,u,...,u) \equiv f(u).$$

Let us introduce:

$$p^n = p_\Delta^n = \sup_i \Big(\frac{u_{i+1}^n - u_i^n}{\Delta x} \Big)_+ \tag{2.2}$$

DEFINITION. A numerical scheme is said to be OSLC (respectively weakly OSLC) consistent if:

$$p_\Delta^n \leq (p(u(0,\cdot))^{-1} + \alpha n \Delta t)^{-1} \tag{2.3}$$

respectively:

$$p_\Delta^n \leq p(u(0,\cdot)) \tag{2.4}$$

Remark: Conditions (2.3) and (2.4) clearly are discrete versions of (1.9) and (1.10).

We have the following result:

PROPOSITION 2. *For the periodic boundary initial value problem, the approximate solutions converge to the correct entropy solution, if either (i) the scheme is OSLC consistent, the initial condition has a finite number of increasing jumps, and the propagation speed of the scheme stays uniformly bounded; or (ii) the scheme is weakly OSLC and the semi-norm of the inital data $p(u(0,\cdot))$ is finite.*

Basically the proof follows from the fact that the OSLC condition enforces both entropy consistency and total variation boundedness. Technical details are given in Appendix 1.

Remark. The same result holds for the Cauchy problem when the initial value $u(0,x)$ is constant for large values of $|x|$.

III. **OSLC consistency of some first order accurate schemes.** Properties (2.3) have been obtained in [4, 7, 12] for the Godunov and the Lax-Friedrichs scheme. It is unclear, to the best of our knowledge, whether or not this property holds for any monotone scheme (or more generally E-scheme, as defined in [8]) It is also clear that the Engquist-Osher scheme is OSLC consistent (as a matter of fact, the EO scheme is the discrete version of the transport-collapse method described in [2], which is clearly OSLC consistent). A more recent result involves the Roe scheme [3]:

PROPOSITION 3. [3]. *The Roe scheme is weakly* OSLC *consistent. When the initial condition satisfies* $p(u(0,\cdot)) < +\infty$, *in particular, when it is smooth, then the approximate solutions converge to the correct physical solution.*

This is a little surprising, since the Roe scheme is usually quoted to be entropy inconsistent! As a matter of fact, the Roe scheme does not violate the entropy condition, provided the semi-norm of the initial data, $p(u(0,\cdot))$ is finite. Otherwise, in particular when the initial data has increasing jumps, the entropy condition can be violated, which explains the apparent contradiction.

IV. **MUSCL schemes and the OSLC.** A priori, it is unclear whether the OSLC is compatible with second, or higher, order accuracy. For example, the E-condition, introduced in [8] to enforce both entropy consistency and total variation boundedness, for any kind of scalar conservation laws (not necessarily convex), is known to be only compatible with 1st order accuracy [8]. An appealing framework in which to discuss higher order accurate schemes, is the one introduced by Van Leer [13] with his MUSCL schemes, a generalization of Godunov's ideas. In a rather abstract way, a MUSCL scheme can be described in terms of the exact evolution (or solution) operator (E_t $t \geq 0$) associated with (1.2) and a projection operator $P_{\Delta x}$

that transforms any given function $v(x)$ into a cell wise polynomial approximation $P_{\Delta x}v(x)$, in such a way that, in each cell, the mean value is preserved:

$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} P_{\Delta x}v(x) = v_i \tag{4.1}$$

where

$$v_i = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v(x)dx \quad \text{and} \quad x_{i-\frac{1}{2}} = (i-\frac{1}{2})\Delta x. \tag{4.2}$$

The numerical scheme is then defined by:

$$u_i^n = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u^n(\cdot)dx, \tag{4.3}$$

where the function $u^n(x)$ is given by:

$$u^n = (E_{\Delta t}P_{\Delta x})^n u(0,\cdot) \tag{4.4}$$

The Godunov scheme corresponds to the simplest choice of the projection operator $P_{\Delta x}$:

$$P_{\Delta x}v(x) = v_i, \text{ for } x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}}, \text{ where } v_i \text{ is given by (4.2).}$$

This scheme uses piecewise constant approximations and is only first order accurate. Therefore Van Leer's idea was to increase the order of accuracy by using polynomial approximations of higher order, when choosing $P_{\Delta x}$.

In order to keep stability and avoid spurious oscillations, $P_{\Delta x}$ is usually required not to increase the total variation:

$$TV(P_{\Delta x}v) \leq TV(v). \tag{4.6}$$

Van Leer found a second order operator $P_{\Delta x}$, using piecewise linear approximations and still compatible with (4.6). In each cell,

$$P_{\Delta x}v(x) = v_i + (x - x_i)s_i \quad \text{where} \quad x_i = i\Delta x, \tag{4.7}$$

and the numerical slope s_i is given by

$$s_i = \phi\Big(\frac{v_{i+1} - v_i}{\Delta x},\ \frac{v_i - v_{i-1}}{\Delta x}\Big), \tag{4.8}$$

with

$$\phi(a,b) = \min\mathrm{mod}(a,b) = \varepsilon \min\{|a|,|b|\}, \varepsilon = \begin{cases} 1 & \text{if } a,b > 0 \\ -1 & \text{if } a,b < 0 \\ 0 & \text{otherwise} \end{cases}$$

Entropy consistency of the resulting scheme is a priori unclear and was proved in [9] (for the method of lines, i.e. the $\Delta t \to 0$ limit case) and in [10] (provided further minor restrictions are added). This scheme is not OSLC consistent, mainly because the corresponding projection operator may increase the one-sided Lipschitz semi-norm p. Conversely we have:

PROPOSITION 4. *A MUSCL scheme designed in such a way that:*

$$p(P_{\Delta x} v) \le p(v) \tag{4.10}$$

is automatically OSLC and therefore convergent.

Proof. The proof is based on the fact that the exact evolution operator E_t satisfies:

$$p(E_{\Delta t} v) \le (p(v)^{-1} + \alpha \Delta t)^{-1} \tag{4.11}$$

as a consequence of (1.9).

Therefore, when (4.10) holds, we get:

$$\begin{aligned} p(u^n) &= p(E_{\Delta t} P_{\Delta x} u^{n-1}) \quad \text{by definition (4.4))} \\ &\le (p(P_{\Delta x} u^{n-1})^{-1} + \alpha \Delta t)^{-1} \quad \text{(by (4.11)),} \\ &\le (p(u^{n-1})^{-1} + \alpha \Delta t)^{-1} \quad \text{(by (4.10)),} \\ &\le \cdots \le (p(u(0,\cdot))^{-1} + \alpha n \Delta t)^{-1} \quad \text{(by induction).} \end{aligned}$$

Then, since, for each i,

$$\frac{u^n_{i+1}-u^n_i}{\Delta x}=\frac{1}{\Delta x}\int_0^{\Delta x}\frac{u^n(x_{i+\frac12}+s)-u^n(x_{i-\frac12}+s)}{\Delta x}ds \quad \text{by definition (4.3))}$$
$$\le p(u^n) \qquad \text{(by definition of } p\text{)},$$

it follows:

$$P^n_\Delta=\sup_i\Big(\frac{u^n_{i+1}-u^n_i}{\Delta x}\Big)_+\le p(u^n)\le (p(u(0,\cdot))^{-1}+\alpha n\Delta t)^{-1} \tag{4.12}$$

which is exactly the discrete OSLC (2.3). This completes the proof.

V. *A fully second order OSLC consistent MUSCL scheme.* Condition (4.10) is not satisfied by the classical MUSCL scheme corresponding to (4.7, 8, 9). Nevertheless,

PROPOSITION 5. *The MUSCL scheme defined by (4.7,8) and:*

$$o(a,b)=\max(a,b) \tag{5.1}$$

is fully second order accurate, satisfies (4.10), therefore is OSLC consistent and convergent.

Proof. Let us first prove (4.10). We have to show, that for any x and y, $x\neq y$, we have

$$\frac{P_{\Delta x}v(x)-P_{\Delta x}v(y)}{x-y}\le p(v). \tag{5.2}$$

Because of definition (4.7), it is enough to check both

$$s_i\le p(v) \qquad \text{and} \tag{5.3}$$

$$\delta_{i+\frac12}\overset{\text{def}}{=}P_{\Delta x}v(x_{i+\frac12}+0)-P_{\Delta x}v(x_{i+\frac12}-0)\le 0. \tag{5.4}$$

hold for each i. [In other words, all discontinuities are decreasing and, in each cell, the slope is bounded from above by $p(v)$].

Let us check (5.3). We have (by definitions (4.8) and (5.1))

$$s_i = \max\Big(\frac{v_{i+1}-v_i}{\Delta x},\ \frac{v_i - v_{i-1}}{\Delta x}\Big)$$
$$= \max\left(\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\Big(\frac{v(x+\Delta x)-v(x)}{\Delta x}\Big)dx,\ \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\Big(\frac{v(x)-v(x-\Delta x)}{\Delta x}\Big)dx\right)$$

(by definition (4.2))

$$\leq \max_{x\neq y}\Big(\frac{v(x)-v(y)}{x-y}\Big) \leq p(v) \quad \text{(by definition (1.1) of } p\text{)}.$$

Thus (5.3) is proven. Let us now look at (5.4). We have

$$\delta_{i+\frac{1}{2}} = \Big(v_{i+1} - \frac{\Delta x}{2}s_{i+1}\Big) - \Big(v_i + \frac{\Delta x}{2}s_i\Big) \quad \text{(by definition (4.8))}$$

Thus

(5.5)

$$\delta_{i+\frac{1}{2}} = (v_{i+1}-v_i) - \frac{\Delta x}{2}\left(\max\Big(\frac{v_{i+2}-v_{i+1}}{\Delta x},\ \frac{v_{i+1}-v_i}{\Delta x}\Big) + \max\Big(\frac{v_{i+1}-v_i}{\Delta x},\ \frac{v_i - v_{i-1}}{\Delta x}\Big)\right)$$

and therefore:

$$\delta_{i+\frac{1}{2}} \leq (v_{i+1}-v_i) - \frac{\Delta x}{2}\Big(\frac{v_{i+1}-v_i}{\Delta x} + \frac{v_{i+1}-v_i}{\Delta x}\Big) = 0$$

which enforces (5.4). This achieves the proof of (5.3-4) and therefore (4.10). So, the scheme is OSLC consistent. Let us now prove it is also second order accurate. To achieve that we shall prove:

(5.6)
$$\frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} dx\big[E_{\Delta t}u(x) - E_{\Delta t}P_{\Delta x}v(x)\big] = O(\Delta x)^2$$

for smooth functions $u(x)$.

We recall

$$v(x) = v_i + \frac{(x-x_i)}{\Delta x}s_i \quad \text{for } x_{i-\frac{1}{2}} < x < x_{i+\frac{1}{2}},$$

where

$$v_i = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u(x)dx$$

and s_i is defined by (4.8), (5.1).

From the divergence theorem, this amounts to showing that:

$$\frac{1}{\Delta t}\int_0^{\Delta t}\left[f(u(x_{i+\frac{1}{2}},t)) - f(u_\Delta(x_{i+\frac{1}{2}},(t))\right]dt \tag{5.7}$$
$$-\frac{1}{\Delta t}\int_0^{\Delta t}\left[f(u(x_{i-\frac{1}{2}},t) - f(u_\Delta(x_{i-\frac{1}{2}},t))\right]dt = O(\Delta x)^2$$

where $u(x,t)$ and $u_\Delta(x,t)$ are the solutions to (1.2) with initial data $u(x)$ and $v(x)$ respectively.

This follows, if in each cell $(x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$:

$$u(x) - v(x) = c(x,x_i)(\Delta x)^2 + O(\Delta x)^3 \tag{5.8a}$$

with

$$|c(x,x_i) - c(x-\Delta x, x_{i-1})| \le K\Delta x. \tag{5.8b}$$

But

$$u(x) - v(x) = \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}[u(x) - u(y)]dy - (x-x_i)s_i$$
$$= (x-x_i)[u_x(x_i) - s_i] + \left[\frac{(x-x_i)^2}{2} - \frac{1}{24}(\Delta x)^2\right]u_{xx}(x_i) + O(\Delta x)^3$$

Thus we must merely show:

$$(u_x(x_i) - s_i) - (u_x(x_{i-1}) - s_{i-1}) = O((\Delta x)^2) \tag{5.9}$$

but a simple calculation shows that

$$u_x(x_i) - s_i = -\frac{1}{2}\Delta x|u_{xx}(y_i)| + O(\Delta x)^2 \tag{5.10}$$

and we are finished.

Remark. We do not recommend this convergent, fully second order scheme for practical calculations when shocks develop, because the limiter defined in (5.1) permits undershoots. This limiter agrees with the usual minmod limiter when a and b are both negative. Otherwise large jumps in the piecewise linear interpolant can be created and the differential equation solution operator may take some time to dissipate the resulting wiggles. See the calculations displayed in Section X below.

For what we believe to be the state-of-the-art in practical non-oscillatory high order accurate shock capturing algorithms, see [6].

VI. *A OSLC consistent method of lines and a second order order convergent Roe scheme.* A MUSCL scheme as described above is not a practical scheme, because it involves the exact computation of (1.2), for $0 \leq t \leq \Delta t$, when the initial value is a piecewise polynomial function. This task is, in general, impossible (except when the initial value is piecewise constant, as for the Godunov scheme). Actually, for schemes such as the ones described in Section IV (through definitions (4.7-8)) it is well known, that it is enough to know the exact solution only for $x = x_{i+\frac{1}{2}}$ and $0 \leq t \leq \Delta t$. This can be obtained through convenient approximations up to any desired accuracy [1], [5].

A different approach consists of considering the semi-discrete scheme obtained when $\Delta t \to 0$ (Δx being fixed), the so-called method of lines. This leads to a stiff system of ordinary differential equations. Then one's favorite ODE solver can be used to get a fully discrete scheme. The main advantage of this approach is the simplicity of the numerical expressions involved in the method of lines. As a matter of fact, the method of lines corresponding to any MUSCL scheme of the

form (4.7-8), can be written as follows (see [9])

$$\frac{du_i}{dt} + \frac{1}{\Delta x}\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) = 0 \tag{6.1}$$

$$f_{i+\frac{1}{2}} = h\left(u_i + \frac{\Delta x}{2}s_i,\ u_{i+1} - \frac{\Delta x}{2}s_i\right) \tag{6.2}$$

where $h(a,b)$ is the Godunov flux associated with f:

$$h(a,b) = h_{\mathrm{God}}(a,b) = \varepsilon \min_{0\le s\le 1} \varepsilon f(a + s(b-a)) \text{ where } \varepsilon = \text{ sign}(b-a) \tag{6.3}$$

In the special case of the OSLC consistent scheme described as above, we have (because of (4.8), (5.1))

$$s_i = \frac{1}{\Delta x}\max(u_{i+1} - u_i,\ u_i - u_{i-1}) \tag{6.4}$$

Since the original scheme is OSLC consistent, its semi-discrete version is also OSLC consistent in the following sense:

$$\frac{u_{i+1}(t) - u_i(t)}{\Delta x} \le (p(u(0,\cdot))^{-1} + \alpha t)^{-1} \tag{6.5}$$

and therefore is fully entropy consistent. In the other hand, we have seen that the OSLC enforces:

$$u_{i+1} - \frac{\Delta x}{2}s_{i+1} \le u_i + \frac{\Delta x}{2}s_i \tag{6.7}$$

(there is no increasing discontinuity in the piecewise linear approximations).

Thus the Godunov flux $h_{\mathrm{God}}(a,b)$ is only used when the entries a, b satisfies $a \ge b$. But, we have the rather obvious result:

LEMMA 2. *Let f be a convex flux, $h_{\mathrm{God}}(a,b)$ the Godunov flux and $h_{\mathrm{Roe}}(a,b)$ the Roe flux:*

$$h_{\mathrm{Roe}}(a,b) = \begin{cases} f(a) & \text{if } (f(b) - f(a))(b-a) \ge 0 \\ f(b) & \text{if } (f(b) - f(a))(b-a) \le 0 \end{cases} \tag{6.8}$$

Then for $a \geq b$, $h_{\text{God}}(a,b) = h_{\text{Roe}}(a,b)$.

Consequently, the OSLC consistent method of lines (6.1), (6.2), can be seen as the second order accurate MUSCL version of the Roe scheme, provided (6.4) is used to define the numerical slopes. This result is quite surprising since, as it has been often noted, the first order accurate Roe scheme is not (fully) entropy consistent.

PROPOSITION 6. *The semi-discrete second order MUSCL Roe scheme, defined by (6.4), is (strongly) OSLC consistent and therefore fully entropy consistent.*

VII. *The Modified equation.* It was pointed out to us by E. Harabetian, that the modified equation corresponding to our new scheme is of interest for the following reason: On one hand, since the scheme is fully second order accurate, the perburbation term in the modified equation involves a third order derivative of the solution and therefore looks like a dispersive term (as in the KDV equation). On the other hand, since the OSLC is enforced, the perturbation term does not create wiggles (as in the KDV equation). After rather obvious computations, it turns out that:

PROPOSITION 7. *The modified equation associated with the second order Roe MUSCL scheme* (6.1), (6.2), (6.4) *is:*

$$u_t + f(u)_x + \delta |u_{xx}|_x = 0 \tag{7.1}$$

with

$$\delta = O(\Delta x^2) \tag{7.2}$$

The study of (7.1), for $\delta > 0$ fixed, is in our opinion an interesting new open problem. Formally the solutions still satisfy the OSLC (1.9). But existence and uniqueness of solutions to (7.1) remain to be investigated.

They should be considered as limits of the smooth solutions of the fourth order parabolic equation:

(7.2) $$u_t + f(u)_x + \delta|u_{xx}|_x = -\varepsilon u_{xxxx},\ \varepsilon \to 0$$

A new entropy condition presumably follows from:

(7.3) u_{xx} has only increasing jumps.

So we have the following:

Conjecture. At least for smooth initial value, the Cauchy problem for (7.1) admits a unique solution $u(t,x)$ such that:

i) $u_{xx}(t,x)$ is in L^1_{loc}

ii) $u_{xxx}(t,x) \geq \gamma > -\infty$, a.e.

Moreover, $u(t,x)$ satisfies the OSLC (1-9).

This conjecture can be extended to:

(7.4) $$u_t + f(u)_x - g(u_{xx})_x = 0$$

for any concave function g (in (7.1), $g(w) = -|w|$)

Existence of travelling waves for (7.1), or (7.4), is a more feasible result, and is now under investigation.

VIII. *Discretization of the modified equation.* The modified equation (7.1), and more generally equation (7.4), can be (semi-)discretized as follows. Let us define a

(first order) numerical flux $g_{i+\frac{1}{2}} = g(w_i, w_{i+1})$ for the concave function g. It could be the Godunov flux:

$$(8.1) \qquad g_{\text{God}}(a,b) = \varepsilon \min_{0\le s\le 1}\big[\varepsilon \cdot g(a + s(b-a))\big],\ \varepsilon = \ \text{sgn}(b-a),$$

The E.O. flux (assuming $g(0) = g'(0) = 0$)

$$(8.2) \qquad g_{\text{EO}}(a,b) = g(b_+) + g(a_-)$$

$$(\text{where } b_+ = \max(o,b),\ a_- = \min(o,b))$$

or the Lax Friedrichs flux of either form:

$$(8.3) \qquad g_{\text{LF}}(a,b) = g\Big(\frac{a+b}{2}\Big) - \frac{\lambda}{2}(b-a)$$

$$\text{or } \frac{g(a)+g(b)}{2} - \frac{\lambda}{2}(b-a)$$

$$\text{where the constant } \lambda \text{ satisfies : } \lambda \ge \max|g'(u)|$$

Then, a method of lines approach can be designed for (7.4):

$$(8.4) \qquad \Delta x \frac{du_i}{dt} + \Delta_-\Big[f\Big(\frac{v_i + v_{i+1}}{2}\Big) - \delta\ g(w_i, w_{i+1})\Big] = 0$$

$$(\text{where } \Delta_- a_i = a_i - a_{i-1})$$

with:

$$(8.5) \qquad w_i = \frac{1}{(\Delta x)^2}(u_{i-1} - 2u_i + u_{i+1})$$

A fully discrete scheme can be designed by replacing $\frac{d}{dt}$ by the Euler forward finite difference:

$$(8.6) \qquad \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{1}{\Delta x}\Delta_-\Big[f\Big(\frac{u_i^n + u_{i+1}^n}{2}\Big) - \delta\ g(w_i^n, w_{i+1}^n)\Big] = 0$$

Then we have, in the case $g(w) = -|w|$.

PROPOSITION 8. *Scheme* (8.6) *satisfies the weak OSLC* (2.3) *under the stability condition:*

$$\left|f'\left(\frac{u_i^n + u_{i+1}^n}{2}\right)\right| \leq \frac{\delta}{\Delta x^2} \leq \frac{\Delta x}{6\Delta t}$$

The proof is given in Appendix 2, in the case when the EO flux is used.

As a consequence, when $\delta = O(\Delta x^2)$, (8.6) is a new OSLC consistent scheme for the conservation law (1.2). By a standard truncation analysis, this scheme is second order accurate in space (first order in time). Condition (8.7) is, then, a standard CFL condition.

IX. *Numerical results.* Let us consider the Burgers equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{9.1}$$

with initial conditions:

$$u_0(x) = \begin{cases} 1 & \text{if } 0 < x < 4 \\ -1 & \text{otherwise} \end{cases} \tag{9.2}$$

The solution at time $t = 4$ has been calculated exactly (Figure 1), by using Van Leer's method (minmod slope limiter, Figure 2), or its "superbee" variant [11] of (Figure 3), and finally the OSLC consistent scheme described above (Figure 4). In each case, the time step was $\Delta t = 0.04$ (25 time steps have been used), the mesh width $\Delta x = 0.2$.

We see the excessive compression of the rarefaction wave using superbee and the expected undershoot at rarefactions using OSLC.

Next we considered a linear equation

$$u_t + u_x = 0 \tag{9.3}$$

with

$$u_0(x) = \sin^2 x, \quad |x - \frac{\pi}{2}| \leq 0 \tag{9.4}$$

$$= 0 \qquad\qquad |x - \frac{\pi}{2}| > \frac{\pi}{2}$$

We graph the solution as it advances one full period.

Figure 5 shows the results using the minmod limiter, Figure 6 uses a compressive almost 3rd order TVD scheme devised in [14], and Figure 7 uses the OSLC scheme. The undershoot in this linear problem is more pronounced, as expected.

REFERENCES

[1] M. Ben-Artzi and J. Falcowitz, *An upwind second-order scheme for compressible duct flows,* Siam J. Sci. Comp., (1986 to appear).

[2] Y. Brenier, *Averaged multivalued solutions for scalar conservation laws,* SINUM V.21, (1984) pp. 1013-1037.

[3] Y. Brenier, *Roe's scheme and entropy solutions for convex scalar conservation laws,* INRIA report #423, (1985) INRIA Rocquencourt, 78150 LeChesnay, France.

[4] J. Goodman and R.J. LeVeque, *A geometric approach to high resolution TVD schemes,* ICASE report #84-55, (1984), SINUM (to appear).

[5] E. Harabetian, *A convergent series expansion for hyperbolic systems of conservation laws,* NASA Contractor Report 172557, ICASE Report #85-13, 1985, AMS Memoirs, to appear.

[6] A. Harten, B. Engquist, S. Osher, and S.R. Chakravarthy, *Uniformly high order accurate essentially non-oscillatory schemes III,* NASA Contractor Report 178101, ICASE Report # 86-22, (1986) submitted to J. Comp. Physics.

[7] O. Oleinik, *Discontinuous solutions of nonlinear differential equations.* Usp. Mat. Nauk. (NS), V.12, (1957), pp. 3-73. English translation in Amer. Math. Soc. Transl. Ser. 2., V.26, pp. 95-172.

[8] S. Osher, *Riemann solvers, the entropy condition, and difference approximations,* SINUM, V.21, (1984), pp. 217-235.

[9] S. Osher, *Convergence of generalized MUSCL schemes,* NASA Langley Contractor Report 172306, (1984), SINUM V.22, (1985), pp. 947-961.

[10] S. Osher and E. Tadmor, *The convergence of difference approximations to scalar conservation laws,* ICASE Report #85-28, NASA CR-172614, submitted to Math. Comp.

[11] P.K. Sweby, *High resolution schemes using flux limiters for hyperbolic conservation laws,* SINUM, V.21, 1984, pp. 995-1011.

[12] E. Tadmor, *The large time behavior of the scalar, genuinely nonlinear Lax-Friedrichs scheme,* Math. Comp. V.43, 1984, pp. 353-368.

[13] B. Van Leer, *Towards the ultimate conservative difference scheme, II. Monotonicity and conservation combined in a second-order scheme,* J. Comp. Phys., V.14, 1974, pp. 361-376.

[14] S. Osher and S. Chakravarthy, *Very high order accurate TVD schemes,* ICASE Report #84-44 (1984).

Appendix 1. (proof of Proposition 2)

Let $u_0(x)$ be a piecewise smooth function defined for $x \in [0, L]$ and L-periodic. Let us denote by N the number of cells in such a way that:

(A1.1) $$L = N\Delta x$$

u_0 has a finite number of discontinuity points $\xi_0 < \xi_1 < \xi_2 < \cdots < \xi_p$. Let us introduce

(A1.2) $$p^*(u_0) = \sup_{i=1,\ldots,p} \sup_{\xi_{i-1}<x\neq y<\xi_i} \left(\frac{u_0(x) - u_0(y)}{x - y}\right)_+$$

which is a finite constant (even if $p(u_0) = +\infty$). Assuming the propagation speed of the scheme to be uniformly bounded by C, let us define:

(A1.3) $$I_n = \{i = 1, \ldots, N;\ \mathrm{dist}(i\Delta x, \Xi) \leq Cn\Delta t + \Delta x\}$$

where $\Xi = \{\xi_0, \xi_1, \ldots, \xi_p\}$. Clearly, the values u_i^n and u_{i+1}^n do not depend on the values of u_0 around Ξ if $i \notin I_n$. Therefore, using the OSLC consistency of the scheme, we get:

(A1.4) $$\frac{u_{i+1}^n - u_i^n}{\Delta x} \leq p^*(u_0) \quad \text{for} \quad i \notin I_n$$

On the other hand, we always have:

$$\frac{u_{i+1}^n - u_i^n}{\Delta x} \leq \frac{1}{\alpha n \Delta t},$$

since the scheme is OSLC consistent.

Thus we have:

(A1.6) $$u_{i+1}^n - u_i^n \leq \Delta x M_i^n, \text{ where } M_i^n = \begin{cases} (\alpha n \Delta t)^{-1} & \text{if } i \in I_n \\ p^*(u_0) & \text{if } i \notin I_n \end{cases}$$

let us now estimate the total variation at time $n\Delta t$:

(A1.7) $$TV_n = \sum_{i=0,\ldots,N-1} |u_{i+1}^n - u_i^n|$$

We have:

$$TV_n \leq \sum |u_{i+1}^n - u_i^n - \Delta x M_i^n| + \sum \Delta x M_i^n$$

and, then (by (A1.6)):

$$TV_n \leq 2 \sum \Delta x M_i^n + \sum (u_{i+1}^n - u_i^n)$$

The second term of the RHS vanishes because of the periodic boundary conditions. Thus:

(A1.8) $$TV_n \leq 2 \sum \Delta x M_i^n = 2\frac{|I_n|\Delta x}{\alpha n \Delta t} + 2(N - |I_n|)\Delta x p^*(u_0)$$

by definition (A1.6).

Clearly, by definition (A1.3), we have:

(A1.9) $$|I_n| \leq \frac{Cn\Delta t}{\Delta x} + 1$$

Therefore:

$$TV_n \leq 2\frac{C}{\alpha} + 2\frac{\Delta x}{\alpha n \Delta t} + 2N\Delta x p^*(u_0)$$

and thus:

(A1.11) $$TV_n \leq 2\frac{C}{\alpha} + 2\frac{\Delta x}{\alpha \Delta t} + 2Lp^*(u_0)$$

since $N\Delta x = L$.

Assuming Δx and Δt to be of the same order of magnitude, it follows that TV_n is uniformly bounded.

Thus, the scheme is total variation stable. Since it is also entropy consistent, the convergence the unique entropy solution follows through standard arguments. This completes the proof of Proposition 2.

Appendix 2 (Proof of Proposition 8)

Let us consider scheme (8.6) in the case when

(A2.1) $$g(w) = -|w| = -(w)_+ - (-w)_+ \quad (\text{with } a_+ = \max(a,0))$$

and the E.O. flux is used:

(A2.2) $$g(u,v) = g_{\text{EO}}(u,v) = -v_+ - (-u)_+.$$

We get

(A2.3) $$0 = \frac{1}{\Delta t}(u_i^{n+1} - u_i^n) + \frac{1}{\Delta x}\Delta_-\Big(f\Big(\frac{u_i^n + u_{i+1}^n}{2}\Big) + \delta\Big(w_{i+1}^n\Big)_+ + \delta\Big(-w_i^n\Big)_+\Big)$$

Let us introduce:

(A2.4) $$P_{i+\frac{1}{2}}^n = \frac{1}{\Delta x}(u_{i+1}^n - u_i^n)$$

we have (by definition (8.5)):

(A2.5) $$w_i^n = \frac{1}{\Delta x}(P_{i+\frac{1}{2}}^n - P_{i-\frac{1}{2}}^n)$$

Using (A2.3), we get:

(A2.6) $$0 = \frac{\Delta x}{\Delta t}(P_{i+\frac{1}{2}}^{n+1} - P_{i+\frac{1}{2}}^n) + \frac{1}{\Delta x}(\phi_1 + \phi_2), \text{ where :}$$

(A2.7) $$\phi_1 = \Delta_-\Big(f\Big(\frac{u_{i+1}^n + u_{i+2}^n}{2}\Big) - f\Big(\frac{u_i^n + u_{i+1}^n}{2}\Big)\Big)$$

and

(A2.8) $$\phi_2 = \delta\Delta_-\Big(\Big(w_{i+2}^n\Big)_+ + \Big(-w_{i+1}^n\Big)_+ - \Big(w_{i+1}^n\Big)_+ - \Big(-w_i^n\Big)_+\Big)$$

Let us rewrite ϕ_2:

(A2.9) $$\begin{aligned}\phi_2 = \delta\Big[&\Big(w_{i+2}^n\Big)_+ + \Big(-w_{i+1}^n\Big)_+ - \Big(w_{i+1}^n\Big)_+ - \Big(-w_i^n\Big)_+ \\ &- \Big(w_{i+1}^n\Big)_+ - \Big(-w_i^n\Big)_+ + \Big(w_i^n\Big)_+ + \Big(-w_{i-1}^n\Big)_+\Big]\end{aligned}$$

Thus

(A2.10) $$\phi_2 \geq \delta\Big[\Big(-w_{i+1}^n\Big)_+ - 2\Big(w_{i+1}^n\Big)_+ - 2\Big(-w_i^n\Big)_+ + \Big(w_i^n\Big)_+\Big]$$

Let us now consider ϕ_1:

(A2.11) $\phi_1 = f(a) - 2f(b) + f(c)$ with

(A2.12) $$a = \frac{1}{2}\Big(u_{i+1}^n + u_{i+2}^n\Big), \quad b = \frac{1}{2}\Big(u_i^n + u_{i+1}^n\Big), \quad c = \frac{1}{2}\Big(u_{i-1}^n + u_i^n\Big)$$

Taylor expansions lead to:

(A2.13) $$\begin{aligned}\phi_1 =& f(b) + f'(b)(a-b) + R_1 \quad - 2f(b)\\ &+ f(b) + f'(b)(c-b) + R_2\end{aligned}$$

where both R_1 and R_2 are positive, since f is convex. Therefore:

(A2.14) $$\phi_1 \geq f'(b)(a - 2b + c)$$

that is, using (A2.12) and definition (8.5)

(A2.15) $$\phi_1 \geq f'\Big(\frac{1}{2}(u_i^n + u_{i+1}^n)\Big)\Big[w_i^n + w_{i+1}^n\Big]\frac{\Delta x^2}{2}$$

Hence, (A2.6-10-15) lead to:

(A2.16) $$\begin{aligned}\frac{\Delta x}{\Delta t}(p_{i+\frac{1}{2}}^{n+1} - p_{i+\frac{1}{2}}^n) \leq& -\frac{1}{\Delta x}\Big(f'\Big(\frac{u_i^n + u_{i-1}^n}{2}\Big)\Big(w_i^n + w_{i+1}^n\Big)\frac{\Delta x^2}{2}\Big)\\ &- \frac{\delta}{\Delta x}\Big[\Big(-w_{i+1}^n\Big)_+ - 2\Big(w_{i+1}^n\Big)_+ - 2\Big(-w_i^n\Big)_+ + \Big(w_i^n\Big)_+\Big]\end{aligned}$$

Let us denote

(A2.17) $$A = p_{i+\frac{3}{2}}, \quad B = p_{i+\frac{1}{2}}, \quad C = p_{i-\frac{1}{2}}$$

Using (A2.4), (A2.16) can be rewritten:

(A2.18)

$$\begin{aligned}\frac{\Delta x}{\Delta t}(B^{n+1} - B^n) \leq& -\frac{1}{\Delta x}f'\Big(\frac{u_i^n + u_{i+1}^n}{2}\Big)\frac{A^n - C^n}{2}\cdot\Delta x\\ &- \frac{\delta}{\Delta x^2}\big[(-A^n + B)_+ - 2(A^n - B^n)_+ - 2(-B^n + C^n)_+ + (B^n - C^n)_+\big]\end{aligned}$$

That is:

(A2.19)

$$B^{n+1} \leq B^n + \nu(A^n - C^n) - \gamma\left[(-A^n + B^n)_+ - 2(A^n - B^n)_+ - 2(-B^n + C^n)_+ + (B^n - C^n)_+\right]$$

where

(A2.20)
$$\nu = -\frac{\Delta t}{\Delta x} f'\left(\frac{u_i^n + u_{i+1}^n}{2}\right), \quad \gamma = \frac{\delta \Delta t}{\Delta x^3}$$

Since:

$$A^n - C^n = (A^n - B^n) + (B^n - C^n)$$
$$= (A^n - B^n)_+ - (B^n - A^n)_+ + (B^n - C^n)_+ - (C^n - B^n)_+,$$

we deduce:

(A2.21)
$$B^{n+1} \leq B^n + (A^n - B^n)_+(2\gamma + \nu) + (B^n - A^n)_+(-\gamma - \nu)$$
$$+ (B^n - C^n)_+(-\gamma + \nu) + (C^n - B^n)_+(2\gamma - \nu)$$

and then:

(A2.22)
$$B^{n+1} \leq B^n$$
$$+ \left[(A^n - B^n)_+ + (C^n - B^n)_+\right](2\gamma + |\nu|)$$
$$+ \left[(B^n - A^n)_+ + (B^n - C^n)_+\right](|\nu| - \gamma)$$

Therefore if:

(A2.23)
$$|\nu| \leq \gamma \text{ and } \gamma \leq \frac{1}{6}$$

we get:

(A2.24)
$$B^{n+1} \leq B^n + \frac{1}{2}\left[(A^n - B^n)_+ + (C^n - B^n)_+\right] \leq \max(A^n, B^n, C^n)$$

According to definitions (A2.17) and (A2.20), this means:

(A2.25)
$$P_{i+\frac{1}{2}}^{n+1} \leq \max(P_{i+\frac{3}{2}}^n,\ P_{i+\frac{1}{2}}^n,\ P_{i-\frac{1}{2}}^n)$$

provided:

(A2.26)
$$\frac{\Delta t}{\Delta x}\left|f'\left(\frac{u_i^n + u_{i+1}^n}{2}\right)\right| \leq \frac{\delta}{\Delta x^2}\frac{\Delta t}{\Delta x} \leq \frac{1}{6}$$

Since (A2.25) implies the weak OSLC condition, the proof of Proposition 8 is complete.

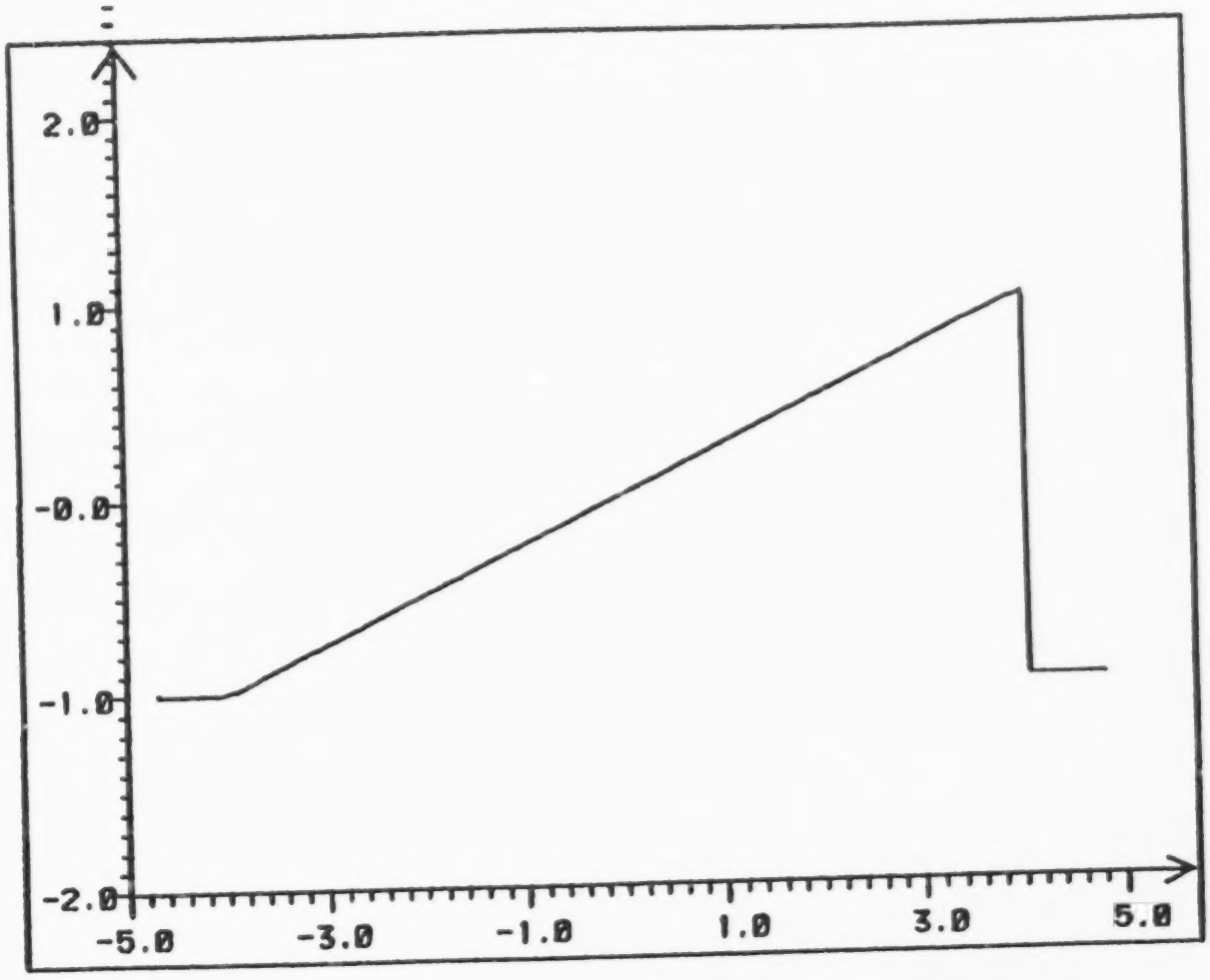

Fig. 1 Exact Solution

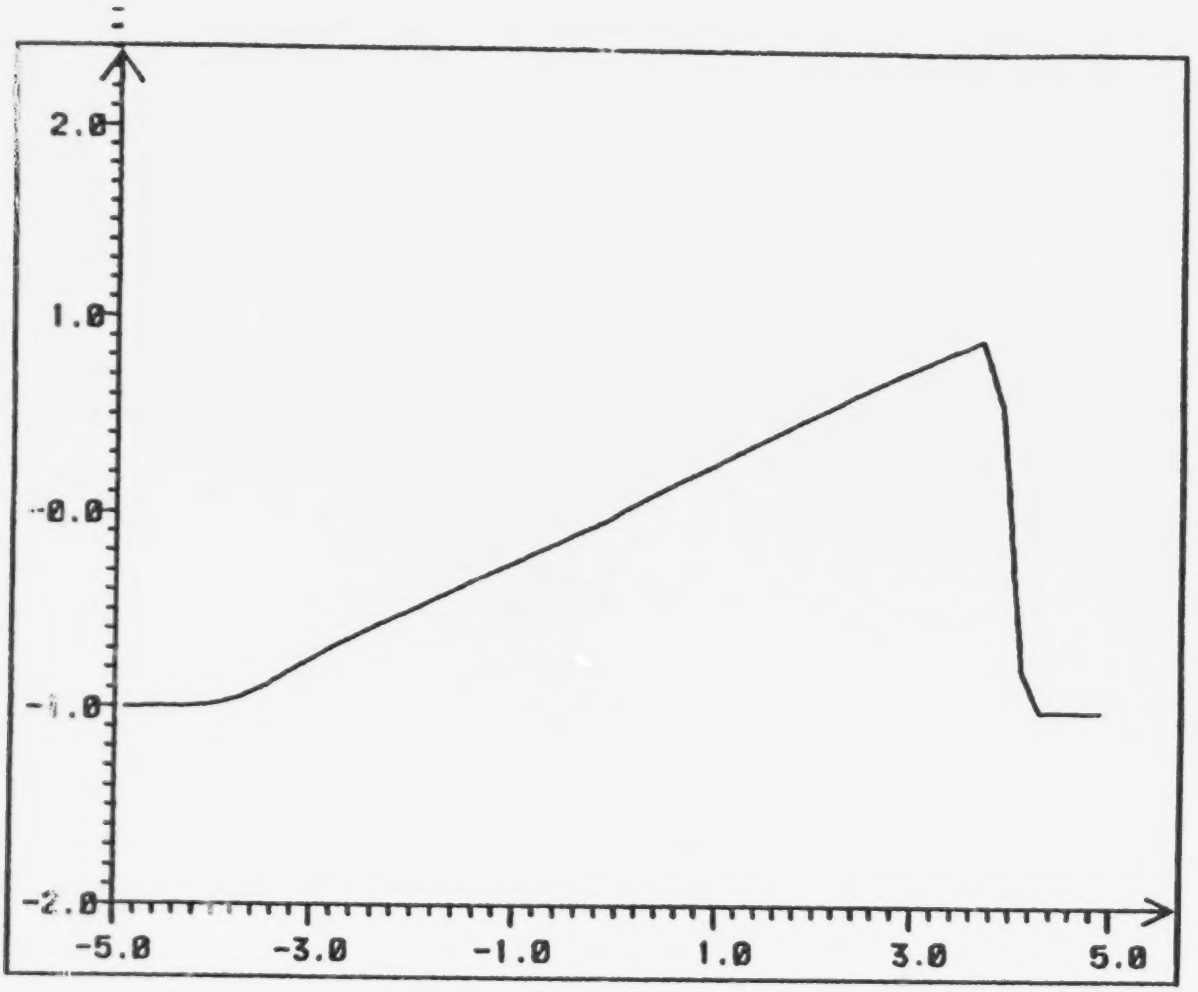

Fig. 2 MUSCL Scheme with
minmod Limiter

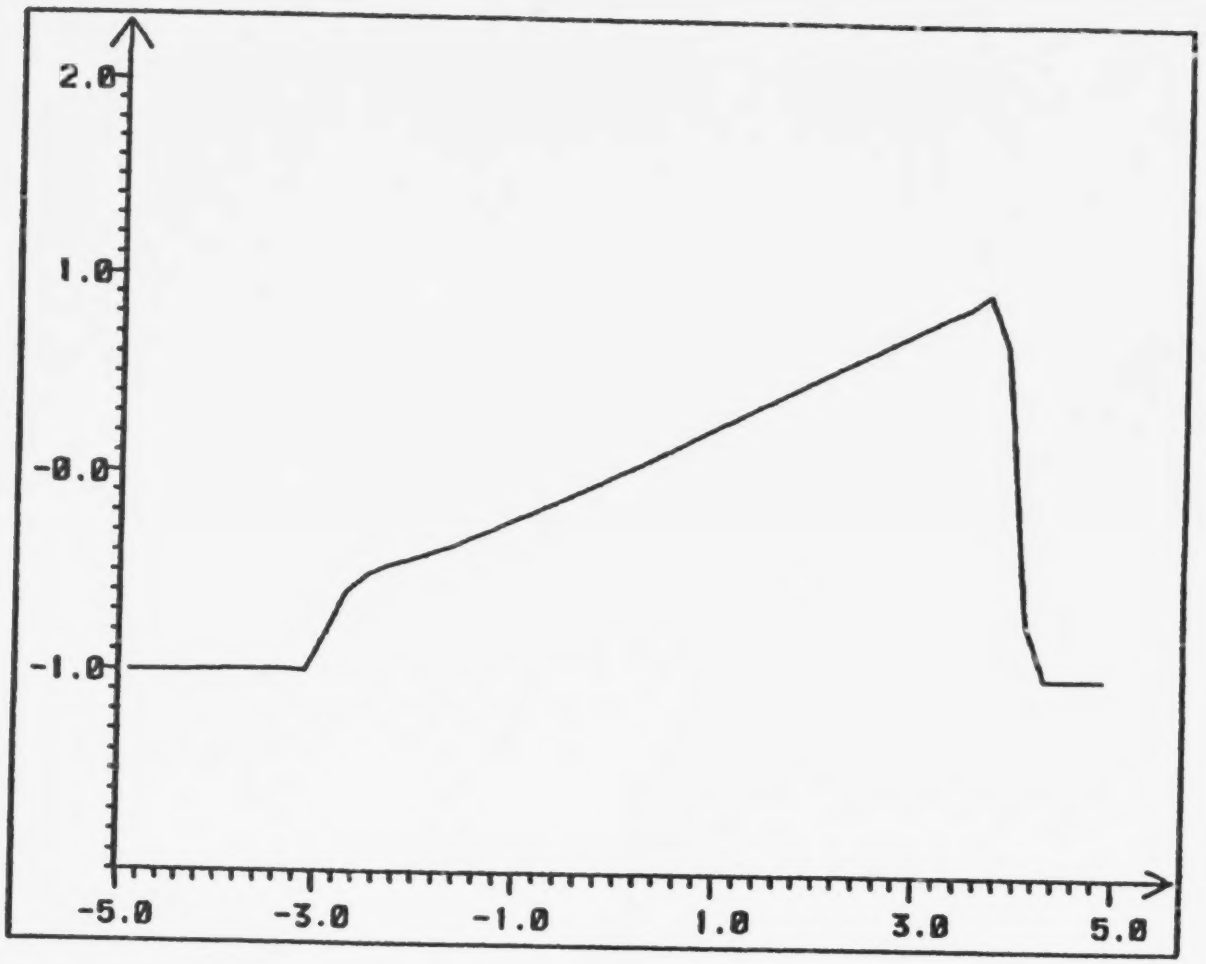

Fig. 3 MUSCL Scheme with
Superbee Limiter

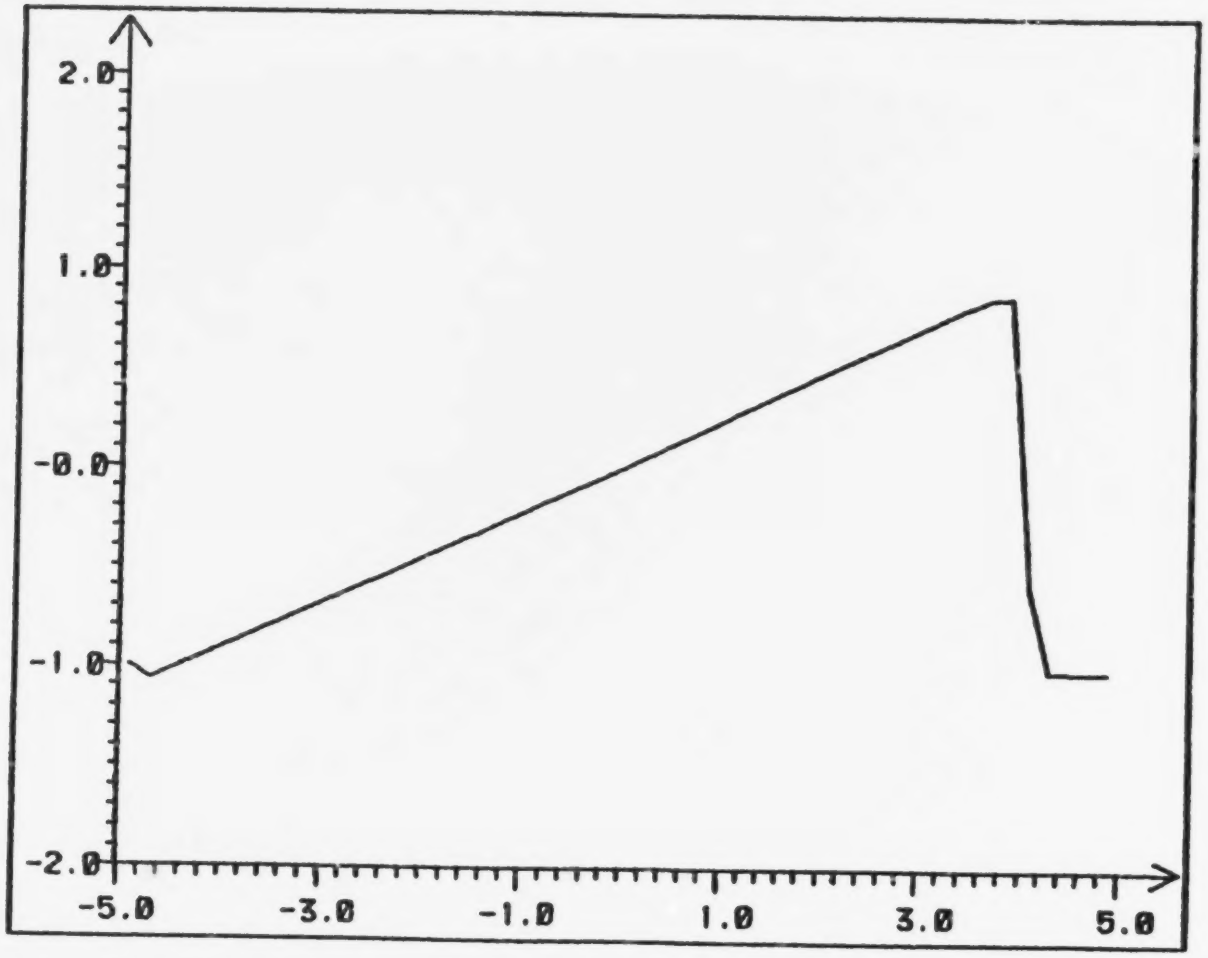

Fig. 4 MUSCL Scheme with
OSLC Limiter

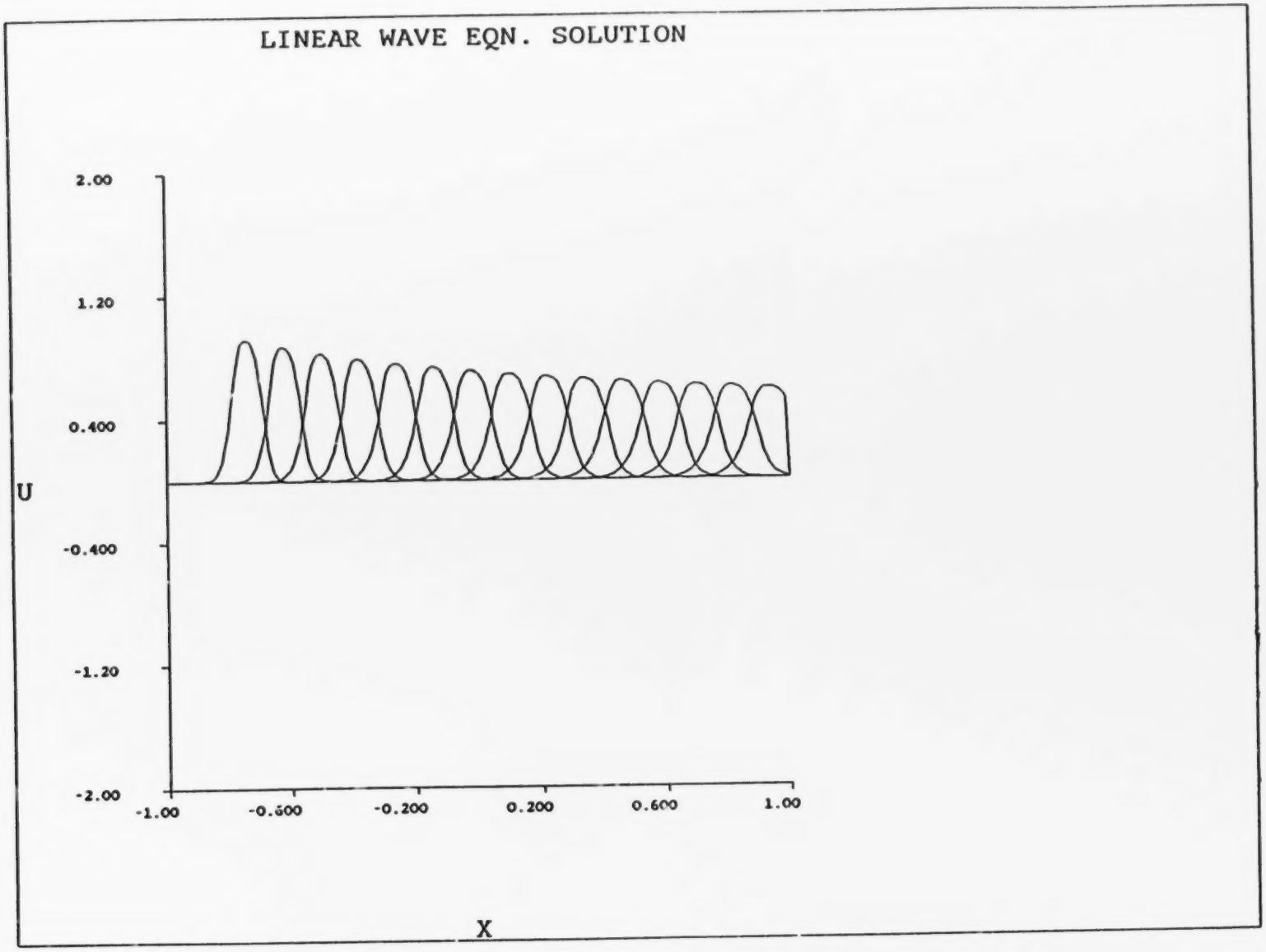

Fig. 5 Minmod Limiter

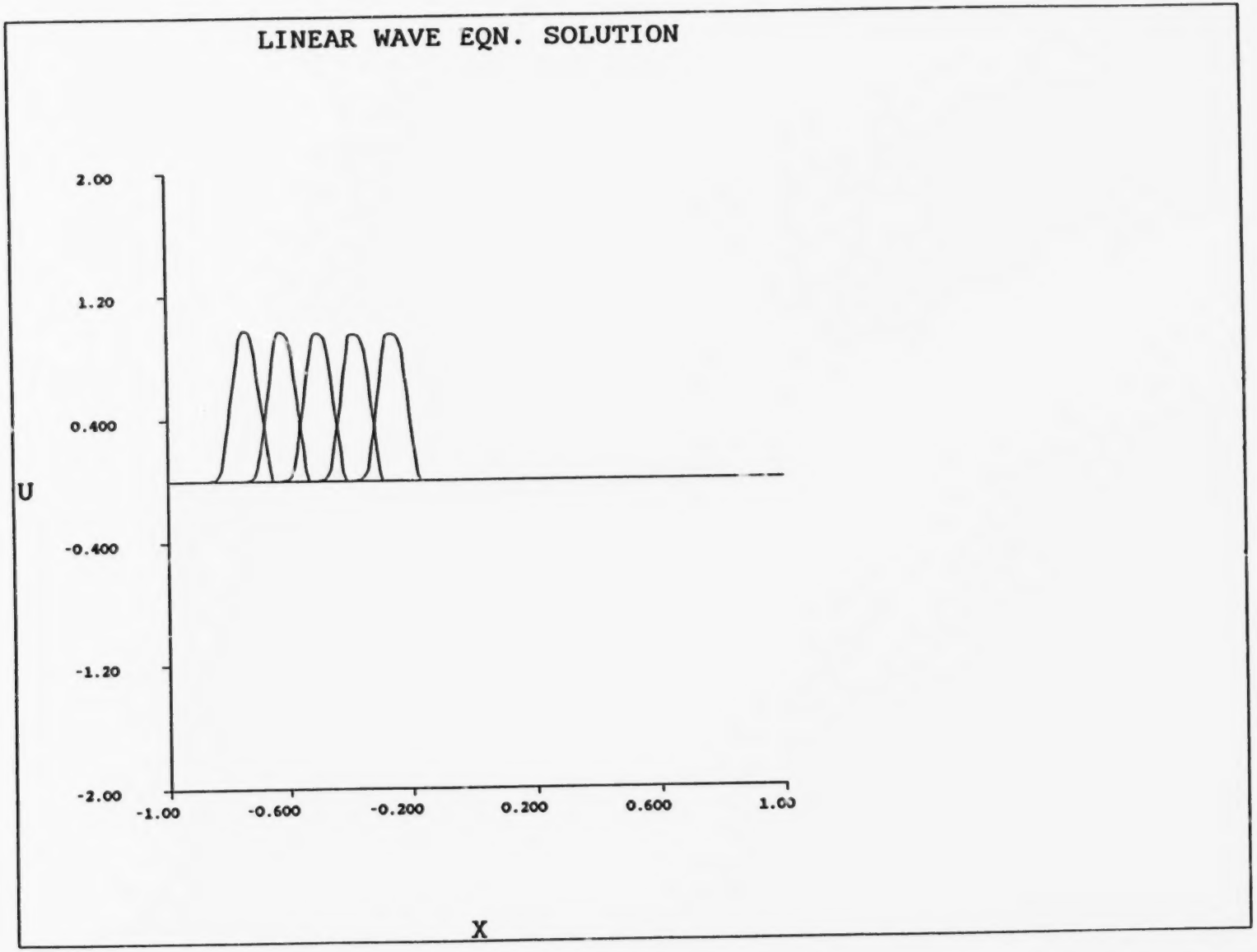

Fig. 6 Compressive 3rd Order TVD Scheme

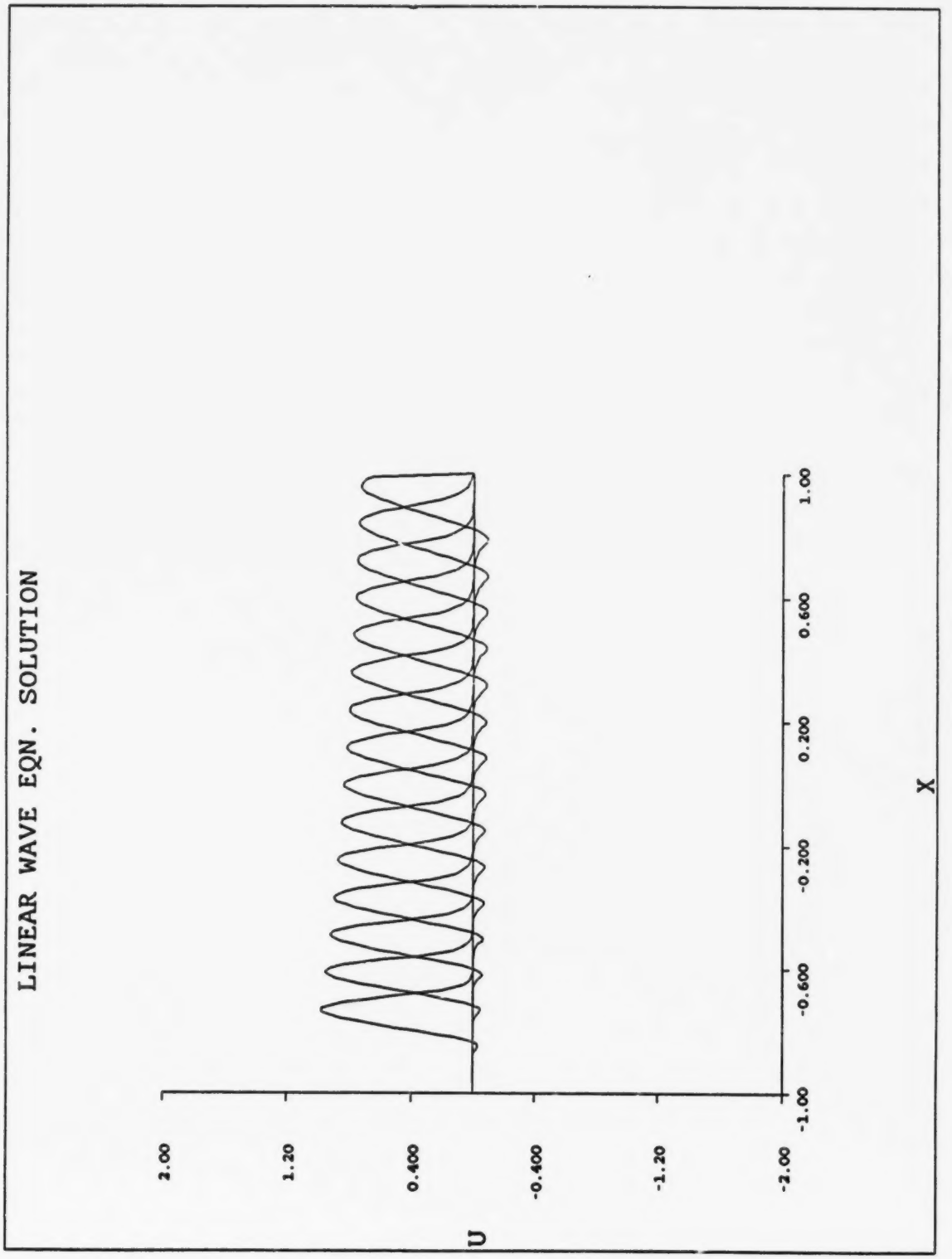

Fig. 7 OSLC Scheme

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16. Abstract

Physical solutions to convex scalar conservation laws satisfy a one-sided Lipschitz condition (OSLC) that enforces both the entropy condition and their variation boundedness. Consistency with this condition is therefore desirable for a numerical scheme and was proved for both the Godunov and the Lax-Friedrichs scheme--also, in a weakened version, for the Roe scheme, all of them being only first order accurate. A new, fully second order scheme is introduced here, which is consistent with the OSLC. The modified equation is considered and shows interesting features. Another second order scheme is then considered and numerical results are discussed.

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